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# On dissipationless decoherence

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## Abstract

We consider a model of environment-induced dissipationless decoherence of a quantum system where the system is coupled to the bath degrees of freedom via the system Hamiltonian itself. We solve exactly for the reduced density operator of the system for an arbitrary spectral density of the thermal bath and also write down an exact master equation in the Lindblad form. We compare and contrast the above results with those obtained by considering the system frequencies to be randomly modulated as in stochastic models. We observe that a coupling to the bath as above necessarily induces a Kerr-like coherent contribution in the reduced dynamics of the system. This Kerr-like term is a reflection of the quantum nature of the bath and cannot be obtained from stochastic models. For the special case of a harmonic oscillator we consider the influence of decoherence and its relation with phase diffusion. Our numerical results exhibit oscillations in the evolution of system variables which overall is a signature of the quantum nature of the environment.

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## 1. Introduction

One of the major interests in quantum statistical mechanics is to unravel the exact quantum feature of dissipation and decoherence of a dynamical system at all time ranges and at all temperatures. Understanding the system–bath interaction to explore dissipation and decoherence was first studied by Feynman and Vernon [1] and was used by Caldeira and Leggett [2] to mimic an open quantum system. The latter authors popularized the quantum Brownian motion model [3] and derived a master equation at high temperatures. System–bath interaction within the weak-coupling regime is also well understood even at very low temperatures, for example, in quantum optics [4]. Some others [5] including Hakim and Ambegaokar [6] have given a master equation for the reduced density matrix of the system at high temperatures, with a non-factorized initial condition. For a free particle, Hu *et al* [7] have provided an exact master equation for a general spectral density of the bath and at arbitrary temperatures, using an influence functional technique.

The high-temperature and long-time limiting behaviour of a quantum system of interest, influenced by its surroundings, is well understood. This behaviour can be pretty well simulated by assuming that the environment introduces a classical noise in the system where the noise can be modelled as a Gaussian–Markov process [8].

We are interested here in a process wherein the system undergoes decoherence but with no dissipation of energy. Such decoherence models are studied by many authors [9] in the Markovian limit or the high-temperature limit or at the level where the bath is acting as a classical noise [5, 10]. A similar dissipationless decoherence model is also studied by Shao *et al* [11]. They have concentrated more on an exact solution of the problem, which we have also given by an operator disentanglement method. After that they solved the problem when the bath is composed of two-level systems instead of harmonic oscillators. Besides, we have asked the following questions: what happens to the decoherence of a quantum system at arbitrary low temperatures and at the ultrashort-time limit? To what extent is the system affected by the quantum nature of the surroundings? How do we obtain the Kubo-oscillator [10] behaviour from a system–heat bath model in the high-temperature limit?

To put our work in proper perspective, let us consider an interaction between a system of interest and the surroundings, i.e. take a total Hamiltonian  $H_T$  as

$$H_T = H_s + H_R + H_{\text{int}}. \quad (1.1)$$

The reservoir Hamiltonian,  $H_R$ , is composed of an infinite number of harmonic oscillators,

$$H_R = \sum_j \hbar \omega_j b_j^\dagger b_j \quad (1.2)$$

with  $[b_i, b_j^\dagger] = \delta_{i,j}$  and assume the interaction is of quantum non-demolition type, satisfying

$$[H_s, H_{\text{int}}] = 0. \quad (1.3)$$

This kind of system–bath interaction is considered by others to obtain the pure decoherence dynamics of the reduced system [9]. Equation (1.3) implies that  $H_{\text{int}}$  is a constant of motion generated by  $H_s$ . For the simplest possible such interaction, we may assume

$$H_{\text{int}} = H_s \sum_j g_j (b_j + b_j^\dagger) \quad (1.4)$$

where  $g_j$  is a  $c$ -number coupling constant. In order to understand the decoherence of various quantum systems, Tameshtit and Sipe [9] have derived a master equation using Born–Markov (BM) approximations as

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_s, \rho] - \gamma_{\text{BM}} (H_s H_s \rho + \rho H_s H_s - 2H_s \rho H_s) \quad (1.5)$$

with

$$\gamma_{\text{BM}} = \frac{\gamma_0 kT}{\hbar} \quad (1.6)$$

where

$$\gamma_0 = \text{Lim}_{\omega \rightarrow 0} 2\pi \frac{|g(\omega)|^2 I(\omega)}{\omega}. \quad (1.7)$$

Here  $I(\omega)$  is the spectral density of the bath and a high-temperature approximation is introduced in concurrence with the Markov approximation.

In what follows we provide an exact solution of the problem in section 2 and compare various classical and quantum features. The BM limit is obtained as a special case of the exact theory. In section 3 we compare the exact result with a classical stochastic model of frequency modulation of the quantum system. Phase diffusion of a harmonic oscillator, which results

from decoherence, is discussed in section 4. In section 5 we show an alternative derivation of the result obtained earlier in section 2, through the introduction of an effective Hamiltonian, and the correspondence with the Kubo-oscillator case is established. The paper is concluded in section 6.

## 2. Exact solution of the model

Here we provide an exact solution for the reduced density operator of the system and, using this solution, derive next a master equation and the concomitant survival probability. For this purpose it is useful to rewrite the Hamiltonian in equation (1.1) as

$$H_T = H_s + \sum_k \hbar \omega_k b_k^\dagger b_k + H_s \sum_k g_k (b_k + b_k^\dagger) + H_s^2 \sum_k \frac{g_k^2}{\hbar \omega_k} \quad (2.1)$$

where  $g_k$  is a dimensionless coupling constant.

Note that we have added a ‘counter-term’, i.e. the last term on the right-hand side of equation (2.1), to make the total Hamiltonian translationally invariant. This can be checked by eliminating the coupling term (i.e. the third term) with the aid of a unitary transformation, defined by

$$U = e^{H_s \sum_k \frac{g_k}{\hbar \omega_k} (b_k^\dagger - b_k)} \quad (2.2)$$

and it is evident that

$$\tilde{H}_T \equiv U H_T U^{-1} = H_s + \sum_k \hbar \omega_k b_k^\dagger b_k \quad (2.3)$$

which is manifestly translationally invariant.

A formal solution of the Liouville equation can be given by

$$\rho(t) = e^{-iH_T t} \rho(0) e^{iH_T t} \quad (2.4)$$

where  $\rho(0)$  is the initial joint density matrix of the system and bath. The latter is assumed to be factorizable, i.e.

$$\rho(0) = \rho^s(0) \rho_B(0) \quad (2.5)$$

where  $\rho_B(0)$  is the equilibrium bath density. The matrix element in the system space is

$$\rho_{nm}(t) = e^{-i(E_n - E_m)t/\hbar} e^{-\frac{i}{\hbar}(E_n^2 - E_m^2) \sum_j \frac{g_j^2}{\hbar \omega_j} t} e^{-iH_n t/\hbar} \rho_B(0) e^{iH_m t/\hbar} \rho_{nm}^s(0) \quad (2.6)$$

where

$$H_n = \sum_k [\hbar \omega_k b_k^\dagger b_k + E_n g_k (b_k + b_k^\dagger)]. \quad (2.7)$$

Taking the trace over the bath operators we obtain

$$\rho_{nm}^s(t) = e^{-i(E_n - E_m)t/\hbar} \text{Tr}_B(\rho_B(0) e^{iH_m t/\hbar} e^{-iH_n t/\hbar}) \rho_{nm}^s(0). \quad (2.8)$$

To simplify the above expression we write

$$H_m^{(k)} = \left[ \hbar \omega_k B_k^\dagger B_k - \frac{E_m^2 g_k^2}{\hbar \omega_k} \right] \quad (2.9)$$

where

$$B_k = b_k + \frac{E_m g_k}{\hbar \omega_k}. \quad (2.10)$$

$B_k$  can be re-expressed as

$$B_k = D^\dagger(\alpha) b_k D(\alpha) \quad (2.11)$$

where  $D(\alpha) = e^{\alpha b_k^\dagger - \alpha^* b_k}$  with  $\alpha = \frac{E_m g_k}{\hbar \omega_k}$ . Therefore, one can write

$$e^{iH_m t/\hbar} = e^{-i\alpha^2 \omega_k t} D^\dagger(\alpha) e^{i\omega_k b_k^\dagger b_k} D(\alpha). \quad (2.12)$$

Now on making use of the properties

$$e^{ix b_k^\dagger b_k} D(\alpha) = D(\alpha e^{ix}) e^{ix b_k^\dagger b_k} \quad (2.13)$$

and

$$D(\alpha) D(\beta) = D(\alpha + \beta) e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} \quad (2.14)$$

we obtain

$$e^{iH_m t/\hbar} = e^{-i\alpha^2(\omega_k t - \text{Sin}(\omega_k t))} D(\alpha(e^{i\omega_k t} - 1)) e^{i\omega_k b_k^\dagger b_k}. \quad (2.15)$$

Therefore one can write

$$e^{iH_m t/\hbar} e^{iH_n t/\hbar} = e^{-i(E_m^2 - E_n^2) \sum_k \frac{g_k^2}{\hbar^2 \omega_k^2} (\omega_k t - \text{Sin}(\omega_k t))} e^{-2(E_m - E_n)^2 \sum_k \frac{g_k^2}{\hbar^2 \omega_k^2} \text{Sin}^2(\omega_k t/2)} e^{\lambda_k b_k^\dagger} e^{-\lambda_k^* b_k} \quad (2.16)$$

where

$$\lambda_k = (E_m - E_n) \frac{g_k}{\hbar \omega_k} (e^{i\omega_k t} - 1). \quad (2.17)$$

We next use the standard result [10]

$$\text{Tr}_B \rho_B \prod_k e^{\lambda_k b_k^\dagger} e^{-\lambda_k^* b_k} = e^{\bar{n}_k |\lambda_k|^2} \quad (2.18)$$

where

$$\bar{n}_k = \frac{1}{2} [\coth(\beta \hbar \omega_k / 2) - 1] \quad (2.19)$$

with

$$\rho_B = \prod_k (1 - e^{-\beta \hbar \omega_k}) e^{-\hbar \omega_k \beta b_k^\dagger b_k}. \quad (2.20)$$

Hence we have the final solution as follows:

$$\rho_{nm}^s(t) = e^{-i(E_n - E_m)t/\hbar} e^{i(E_n^2 - E_m^2)\eta(t)} e^{-(E_n - E_m)^2 \gamma(t)} \rho_{nm}^s(0) \quad (2.21)$$

where

$$\eta(t) = - \sum_k \frac{g_k^2}{\hbar^2 \omega_k^2} \text{Sin}(\omega_k t) \quad (2.22)$$

and

$$\gamma(t) = 2 \sum_k \frac{g_k^2}{\hbar^2 \omega_k^2} \text{Sin}^2(\omega_k t/2) \coth(\beta \hbar \omega_k / 2). \quad (2.23)$$

Note that the above result does not depend on the structure of  $H_s$ , the system Hamiltonian, unlike the quantum Brownian case.

We now assume the bath oscillators to be continuously distributed with a spectral density  $G(\omega)$ , so that for an arbitrary function  $f(\omega)$  the continuum limit implies

$$\sum_k \frac{g_k^2}{\hbar^2} f(\omega_k) \rightarrow \int_0^\infty d\omega G(\omega) f(\omega). \quad (2.24)$$

For the special case of ohmic spectral density [3]

$$G(\omega) = \frac{\gamma_0}{\pi} \omega e^{-\omega/\omega_c} \quad (2.25)$$

where  $\gamma_0$  has the dimension of  $1/(\text{energy})^2$ .  $\gamma_0$  and  $\omega_c$  are the two bath parameters characterizing the quantum noise, which correspond to the strength of the noise and the inverse of the correlation time respectively.

One can find the rate equation equivalent to the above equation of  $\rho_{n,m}^s(t)$  as

$$\frac{d\rho_{nm}^s(t)}{dt} = -i\omega_0(E_n - E_m)\rho_{nm}^s(t) + i\dot{\eta}(t)(E_n^2 - E_m^2)\rho_{nm}^s(t) - \dot{\gamma}(t)(E_n - E_m)^2\rho_{nm}^s(t). \quad (2.26)$$

Similarly, a master equation can be written in the operator form as

$$\dot{\rho}^s = -i[H_s, \rho^s] + i\dot{\eta}(t)[H_s^2, \rho^s] - \dot{\gamma}(t)(H_s^2\rho^s - 2H_s\rho^s H_s + \rho^s H_s^2). \quad (2.27)$$

We now discuss implications of these results for the ohmic bath. First note that for the above spectral density,  $\eta(t)$  can be calculated as

$$\eta(t) = -\frac{\gamma_0}{\pi} \tan^{-1} \omega_c t. \quad (2.28)$$

On the other hand, the expression for  $\dot{\gamma}(t)$  cannot be obtained analytically for all times and at arbitrary temperature and we perform numerical integration for exact evaluation. At high temperature, we consider the Markov limit by replacing  $\coth(\frac{\hbar\omega}{2kT})$  by  $\frac{2kT}{\hbar\omega}$ . Then the expression of  $\dot{\gamma}(t)$  will be equal to

$$\dot{\gamma}(t) = \frac{2kT\gamma_0}{\pi\hbar} \tan^{-1}(\omega_c t). \quad (2.29)$$

Hence, we may obtain various limiting cases of decoherence. For example, in the classical stochastic noise process Kubo [10] had coined the terms slow-modulation (SM) and fast-modulation (FM) limits. In a similar spirit one can investigate two cases. (a)  $\omega_c t \gg 1$ , so that  $\tan^{-1} \omega_c t \approx \pi/2$ . In this limit  $\dot{\gamma}(t)$  assumes the BM value,  $\gamma_{\text{BM}}$ , and we obtain

$$\gamma_{\text{BM}} = \frac{\gamma_0 kT}{\hbar}. \quad (2.30)$$

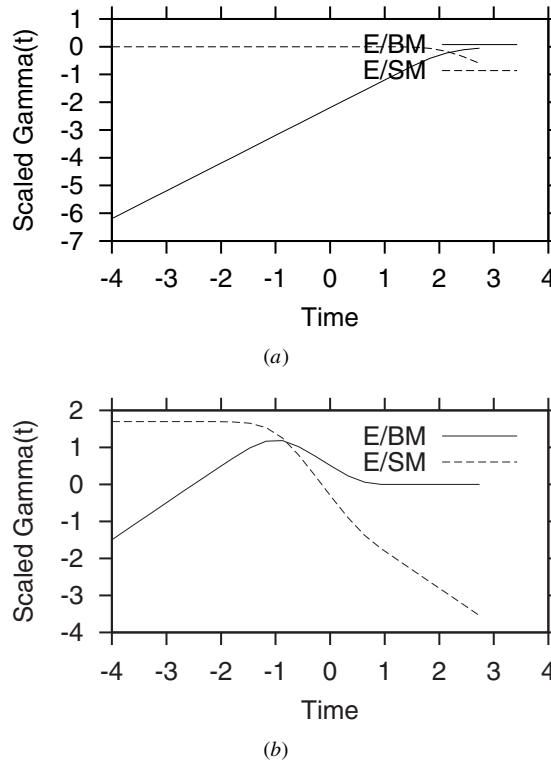
This is used in [9].

(b) When  $\omega_c t \ll 1$  one can write  $\tan^{-1} \omega_c t \approx \omega_c t$ . Therefore, in the SM limit  $\dot{\gamma}(t)$  can be written as

$$\gamma_{\text{SM}} = \frac{2}{\pi} \gamma_{\text{BM}} \omega_c t. \quad (2.31)$$

In what follows we compare the time dependence of the exact  $\dot{\gamma}(t)$  with the two cases of Markovian limiting values as described above. In figure 1 we have plotted a scaled  $\dot{\gamma}(t)$  as  $\text{gamma}(t) = \log_{10} \frac{\dot{\gamma}(t)}{\gamma_{\text{BM}}} \equiv E/\text{BM}$ , or  $\text{gamma}(t) = \log_{10} \frac{\dot{\gamma}(t)}{\gamma_{\text{SM}}} \equiv E/\text{SM}$  with a scaled time,  $\log_{10}(\omega_0 t)$ , for two different values of  $\frac{\omega_c}{\omega_0} = 0.01$  (figure 1(a)) and 10.0 (figure 1(b)) at fixed low temperature  $\frac{kT}{\hbar\omega_0} = 0.1$ . Here  $\omega_0$  is an arbitrary scaling frequency of the system. One finds that at early time  $E/\text{SM} = 0$  and this quantity goes to  $-\infty$  as the scaled time  $t\omega_0$  goes to  $+\infty$ , but  $E/\text{BM}$  increases linearly with time at early times and then it assumes the value zero, after a time that depends on  $\omega_c$ . When  $\frac{\omega_c}{\omega_0}$  increases, the exact  $\dot{\gamma}(t)$  assumes the  $\gamma_{\text{BM}}$  value quite rapidly and deviates from the  $\gamma_{\text{SM}}$  value. With increase in temperature this picture does not change much except that the exact  $\dot{\gamma}(t)$  assumes the  $\gamma_{\text{BM}}$  value even more rapidly at moderate values of  $\frac{\omega_c}{\omega_0}$ .

This implies that at very early times  $\dot{\gamma}(t)$  is proportional to time, and then after some incubation period, depending on the correlation time of the bath,  $\omega_c^{-1}$ ,  $\dot{\gamma}(t)$  assumes the BM



**Figure 1.** Scaled  $\dot{\gamma}(t)$  is plotted as  $\text{Gamma}(t) = \log_{10} \frac{\dot{\gamma}(t)}{\dot{\gamma}_{\text{BM}}} \equiv E/\text{BM}$ , or  $\text{Gamma}(t) = \log_{10} \frac{\dot{\gamma}(t)}{\dot{\gamma}_{\text{SM}}} \equiv E/\text{SM}$  with a scaled time,  $\log_{10}(\omega_0 t)$ , for two different values of  $\frac{\omega_c}{\omega_0} = 0.01$  (figure 1(a)) and 10.0 (figure 1(b)) at a fixed low temperature  $\frac{kT}{\hbar\omega_0} = 0.1$ . Here  $\omega_0$  is an arbitrary scaling frequency of the system.

result. For not so short correlation times and low temperatures, the  $\dot{\gamma}(t)$  value oscillates before arriving at the BM value,  $\dot{\gamma}_{\text{BM}}$ . These details are not shown in the figures. Note that this result does not depend on the structure of  $H_s$  at all, unlike the quantum Brownian case, i.e. for non-diagonal coupling with the bath.

To probe the decoherence behaviour in the special case when  $H_s$  represents the Hamiltonian of a harmonic oscillator we have calculated the survival probability  $P(t) = \text{Tr}(\rho^s(0)\rho^s(t))$ , where  $\rho^s(t)$  is the reduced density of the system, taking the initial density matrix of coherent state,  $\rho^s(0) = |\alpha\rangle\langle\alpha|$ .  $P(t)$  is therefore written as

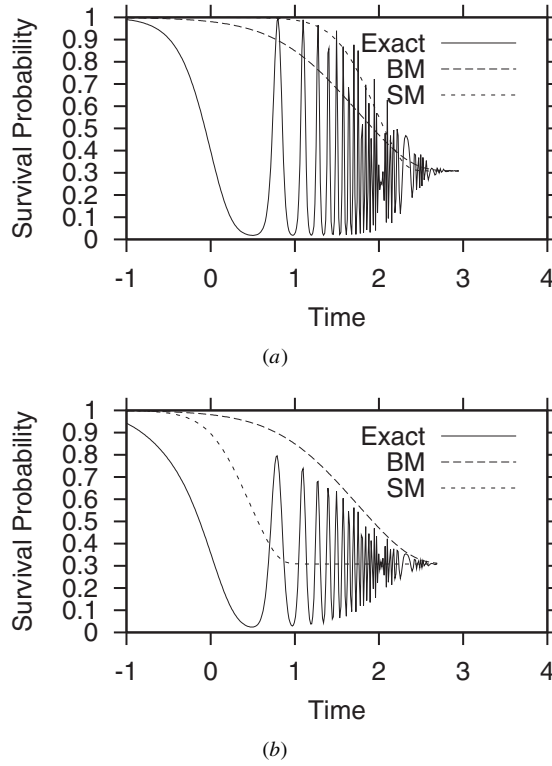
$$P(t) = \sum_{m,n} |\rho_{m,n}^s(0)|^2 U_{n,m}(t) \quad (2.32)$$

where

$$U_{n,m}(t) = e^{-it(n-m)\omega_0} e^{i\hbar^2\omega_0^2[(n^2-m^2)+(n-m)]\eta(t)} e^{-\hbar^2\omega_0^2(n-m)^2\gamma(t)} \quad (2.33)$$

and for a harmonic oscillator  $E_n = (n + \frac{1}{2})\hbar\omega_0$ . Note that the term  $e^{i\hbar^2\omega_0^2(n-m)\eta(t)}$  in  $U_{nm}(t)$  depends on the zero-point energy of the harmonic oscillator.

In figure 2 we have shown the plots of survival probability for exact dynamics, BM dynamics and in the SM limit, at fixed low temperature  $\frac{kT}{\hbar\omega_0} = 0.1$ . We have taken  $(\hbar\omega_0)^2\gamma_0 = 0.1$  and  $\alpha = 1.0$ . In figure 2(a),  $\frac{\omega_c}{\omega_0} = 0.01$ , the exact dynamics is closer to



**Figure 2.** Survival probability, for exact dynamics (exact), by the BM dynamics and due to decoherence in the SM limit, is plotted with scaled time,  $\log_{10}(\omega_0 t)$ . In figure 2(a),  $\frac{\omega_c}{\omega_0} = 0.01$ , the exact dynamics is closer to BM than SM, but in figure 2(b),  $\frac{\omega_c}{\omega_0} = 10.0$ , the exact dynamics is closer to SM than BM.

BM than SM, but in figure 2(b),  $\frac{\omega_c}{\omega_0} = 10.0$ , the exact dynamics is closer to SM than BM. This is apparent from the dependence of  $\gamma_{SM}$  on  $\omega_c$ . Here we find oscillatory decay in the exact dynamics before the survival probability saturates to its asymptotic value  $\sum_m |\rho_{mm}^s(0)|^2$ . At higher temperatures at a certain moderate value of  $\omega_c$ , these three curves coincide because the decoherence rate increases very fast with increase in time. There are more oscillations at lower temperatures. This oscillatory behaviour is a signature of the quantum nature of the bath.

### 3. Master equation for a system undergoing random frequency modulation

Here we consider an arbitrary quantum system undergoing random frequency modulation by a classical stochastic process. We construct a master equation for the random frequency modulation case and compare with the case of the system–bath model in the earlier section.

The equation of motion for the density matrix of a quantum system in the eigenbasis of the Hamiltonian,  $\{|E_n\rangle\}$ , which can be given by

$$\dot{\rho}_{nm} = -i\Omega_{nm}\rho_{nm} \tag{3.1}$$

where  $\rho_{nm} = \langle E_n | \rho | E_m \rangle$  and  $\Omega_{nm} = (E_n - E_m)/\hbar$ . Let us assume that the transition frequencies  $\Omega_{nm}$  undergo a random modulation by a stationary Gaussian noise, i.e.  $\Omega_{nm}$  is



now time dependent, and say

$$\Omega_{nm}(t) = \omega_{nm}[1 + \beta(t)] \quad (3.2)$$

so that equation (3.1) becomes

$$\dot{\rho}_{nm} = -i\omega_{nm}(1 + \beta(t))\rho_{nm} \quad (3.3)$$

with

$$\omega_{nm} = \frac{E_n - E_m}{\hbar} \quad (3.4)$$

$$\langle \beta(t) \rangle = 0 \quad (3.5)$$

and

$$\langle \beta(t)\beta(t') \rangle = \Psi(t - t'). \quad (3.6)$$

One can straightforwardly solve [4] for the averaged density matrix as

$$\langle \rho_{nm}(t) \rangle = e^{-i\omega_{nm}t} e^{-\frac{\omega_{nm}^2}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 \Psi(t_1 - t_2)} \rho_{nm}(0) \quad (3.7)$$

where the symbol  $\langle \dots \rangle$  means the average over the classical stochastic variable.

The above equation can be cast into an equivalent master equation as

$$\langle \dot{\rho}_{nm} \rangle = -i\omega_{nm} \langle \rho_{nm} \rangle - (E_n - E_m)^2 \gamma_{cl}(t) \langle \rho_{nm} \rangle \quad (3.8)$$

where we have defined

$$\gamma_{cl}(t) = \frac{1}{2\hbar^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \Psi(t_1 - t_2). \quad (3.9)$$

Note that, like  $\gamma(t)$  in the quantum case,  $\gamma_{cl}(t)$  too is system independent. The master equation can be written in the operator form as

$$\langle \dot{\rho} \rangle = -i[H_s, \langle \rho \rangle] - \dot{\gamma}_{cl}(t)(H_s^2 \langle \rho \rangle - 2H_s \langle \rho \rangle H_s + \langle \rho \rangle H_s^2). \quad (3.10)$$

The remarkable point is that it is generally valid for any structure of  $\Psi(t_1 - t_2)$ . Only the classical stochastic modulation has to be Gaussian if we are to obtain the same structure for the master equation as in the quantum BM case.

#### 4. Diffusion: harmonic oscillator system

Here we have analysed the result of the last two sections for the case of a harmonic oscillator as the system of interest. The master equations in the two cases can be cast into the respective equivalent equations of quasiprobability distribution functions in a quasiclassical phase space. Then it is shown that the behaviour of diffusion in the two cases is different.

Considering a harmonic oscillator as the system of interest, i.e.

$$H_s = \hbar\omega_0(a^\dagger a + 1/2) \quad (4.1)$$

the master equation for the quantum system–bath case in section 2 becomes

$$\dot{\rho}^s = -i\omega_0[a^\dagger a, \rho^s] + i\dot{\eta}(t)[(a^\dagger a)^2 + a^\dagger a, \rho^s] - \dot{\gamma}(t)[(a^\dagger a)^2 \rho^s - 2a^\dagger a \rho^s a^\dagger a + \rho^s (a^\dagger a)^2]. \quad (4.2)$$

Thus, in the master equation where we have considered the bath to be quantum mechanical in nature, there is an additional Kerr term. On the other hand, the classical equation, where the environment is modelled as a classical stochastic process but the system is quantum mechanical, has the structure

$$\dot{\rho}^s = -i\omega_0[a^\dagger a, \rho^s] - \hbar^2 \omega_0^2 \dot{\gamma}_{cl}(t)[(a^\dagger a)^2 \rho^s - 2a^\dagger a \rho^s a^\dagger a + \rho^s (a^\dagger a)^2]. \quad (4.3)$$

It is well understood that the master equation for the classical case can be cast into the form of the equation of motion of the Glauber–Sudarshan [12]  $P$ -distribution function. Using

$$\rho = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha| \tag{4.4}$$

the master equation of the classical case can be cast into the form of the diffusion equation,

$$\frac{\partial P}{\partial t} = -\omega_0 \left( \frac{\partial}{\partial \alpha} \alpha - \frac{\partial}{\partial \alpha^*} \alpha^* \right) P - \dot{\gamma}_{\text{cl}}(t) \hbar^2 \omega_0^2 \left( \frac{\partial}{\partial \alpha} \alpha - \frac{\partial}{\partial \alpha^*} \alpha^* \right)^2 P. \tag{4.5}$$

In polar co-ordinates  $\alpha = r e^{i\theta}$ , the above equation becomes

$$\frac{\partial P}{\partial t} = 2i\omega_0 \frac{\partial P}{\partial \theta} + 4\dot{\gamma}_{\text{cl}}(t) \hbar^2 \omega_0^2 \left( \frac{\partial^2 P}{\partial \theta^2} \right). \tag{4.6}$$

The second term represents time-dependent diffusion on a circle. This is the reason why the randomly modulated oscillator is taken to model the output of a laser far above threshold, which ideally undergoes only phase fluctuations and no amplitude fluctuation [4].

Furthermore one can also look upon  $\theta$  as the classical phase of the optical field. In this spirit therefore one can say that, just as the randomly modulated oscillator describes the diffusion of the classical phase of the single-mode light field, the quantum nondemolition Hamiltonian [9] describes diffusion of the ‘quantum’ phase [13] of the light field. The Pegg–Barnett analysis [13] of the quantum phase of the exact quantum master equation is interesting to investigate and is left for future work. However it should be noted that the equation of motion for the  $P$ -distribution corresponding to the quantum mechanical master equation is not a pure diffusion equation because of the presence of the additional Kerr-like term.

### 5. Effective Hamiltonian with quantum stochastic terms:

Here we would like to show that the model of the quantum system–bath case that we have considered in section 2 can equivalently be cast into the form of a randomly modulated quantum system where the random modulation is a quantum stochastic process. For this purpose we have assumed that the system is a harmonic oscillator and subsequently we have constructed an effective Hamiltonian. In the high-temperature limit, the effective quantum stochastic process gives the same result as a classical stochastic process and consequently we could provide the microscopic basis of Kubo’s result as a limiting case of our exact result.

From the Hamiltonian

$$H = \hbar\omega_0(a^\dagger a + 1/2) + \sum_k \hbar\omega_k b_k^\dagger b_k + \sum_k g_k (b_k^\dagger + b_k) \hbar\omega_0 (a^\dagger a + 1/2) + [\hbar\omega_0 (a^\dagger a + 1/2)]^2 \sum_k \frac{g_k^2}{\hbar\omega_k} \tag{5.1}$$

one can write the Heisenberg equation of motion for the bath operators

$$\dot{b}_k = -i\omega_k b_k - ig_k \omega_0 (a^\dagger a + 1/2). \tag{5.2}$$

Noting that  $a^\dagger a$  is a constant of motion the above equation can be solved to give

$$b_k(t) = e^{-i\omega_k t} b_k(0) - \frac{g_k}{\omega_k} (1 - e^{-i\omega_k t}) \omega_0 (a^\dagger a + 1/2). \tag{5.3}$$

Similarly, the Heisenberg equation of motion for the system is

$$\dot{a} = -i\omega_0 a - i \sum_k g_k \omega_0 (b_k(t) + b_k^\dagger(t)) a + i2(\hbar\omega_0)^2 (a^\dagger a + 1) a \sum_k \frac{g_k^2}{\hbar\omega_k}. \tag{5.4}$$

Using the formal solutions of the bath operators we obtain

$$\dot{a} = -i\omega_0 a - i\omega_0 B(t)a + i\dot{\eta}(t)2\hbar^2\omega_0^2(a^\dagger a + 1/2)a \quad (5.5)$$

where we have defined

$$B(t) = \sum_k g_k (e^{-i\omega_k t} b_k(0) + e^{i\omega_k t} b_k^\dagger(0)) \quad (5.6)$$

and

$$\dot{\eta}(t) = - \sum_k \frac{g_k^2}{\hbar^2 \omega_k} \cos \omega_k t. \quad (5.7)$$

For a thermal bath one can find that

$$\langle B(t) \rangle = 0 \quad (5.8)$$

$$\langle B(t_1) B(t_2) \rangle = \sum_k g_k^2 [e^{-i\omega_k(t_1-t_2)} (1 + \bar{n}(\omega_k)) + e^{i\omega_k(t_1-t_2)} \bar{n}(\omega_k)]. \quad (5.9)$$

In the high-temperature limit, i.e.  $\bar{n}(\omega) \approx \frac{1}{\beta\hbar\omega}$ , and then taking the continuum bath density modes, we obtain

$$\langle B(t_1) B(t_2) \rangle = \frac{2\hbar}{\beta} \int_0^\infty d\omega \frac{G(\omega)}{\omega} \cos \omega(t_1 - t_2) \equiv \Psi(t_1 - t_2) \quad (5.10)$$

as in the stochastic model.

For the ohmic bath [3]  $G(\omega) = \frac{\gamma_0}{\pi} \omega e^{-\omega/\omega_c}$  one can show that

$$\langle B(t_1) B(t_2) \rangle = \frac{2/\omega_c}{\frac{1}{\omega_c^2} + (t_1 - t_2)^2} \frac{\hbar\gamma_0}{\pi\beta} \quad (5.11)$$

which in the limit  $\omega_c \rightarrow \infty$  becomes

$$\langle B(t_1) B(t_2) \rangle = \frac{4\hbar\gamma_0}{\beta} \delta(t_1 - t_2). \quad (5.12)$$

If we take a Lorentzian cut-off function, i.e.

$$G(\omega) = \frac{\gamma_0}{\pi} \omega \frac{\omega_c^2}{\omega^2 + \omega_c^2} \quad (5.13)$$

then

$$\langle B(t_1) B(t_2) \rangle = \frac{4\hbar\gamma_0}{\beta} \omega_c e^{-\omega_c|t_1-t_2|} \quad (5.14)$$

and this will correspond to the Kubo oscillator case [10]. Thus using the later  $G(\omega)$  in equation (5.13) and in the high-temperature limit we obtain

$$\dot{\gamma}(t) = \gamma_0 \frac{2kT}{\hbar} [1 - e^{-\omega_c t}] \quad (5.15)$$

which gives the same result of SM and FM limits as given in equations (2.28) and (2.29) respectively. This establishes the exact microscopic picture of Kubo's result [10].

Thus we have shown that in the high-temperature limit the correlations of  $B$  operators are described by a classical stationary Gaussian process. The same cannot be done for low  $T$  because the underlying probability distribution would not be positive definite.

Note that equation (5.5) follows from the effective Hamiltonian as

$$\dot{a} = \frac{i}{\hbar} [H_{\text{eff}}(t), a] \quad (5.16)$$

where

$$H_{\text{eff}}(t) = \hbar\omega_0(a^\dagger a + \frac{1}{2})(1 + B(t)) - 2\dot{\eta}(t)(a^\dagger a + \frac{1}{2})^2 \hbar^2 \omega_0^2. \quad (5.17)$$

If one now makes the ansatz that the  $H_{\text{eff}}$  governs the time evolution of the reduced density operator of the harmonic oscillator, i.e.

$$\dot{\rho}^s = -\frac{i}{\hbar}[H_{\text{eff}}(t), \rho^s] \quad (5.18)$$

one can straightforwardly arrive at the general solution as given in section 3.

## 6. Conclusion

We have given an exact solution to the model of environment-induced dissipationless decoherence. The equation of motion of the reduced density matrix is in the Lindblad [14] form where the co-efficients are time dependent. Thereby we could provide two different approaches to look at the dissipationless decoherence problem. (i) We could obtain a microscopic point of view where it is shown how the quantum nature of the bath influences the decoherence process. (ii) The same process is understood as a quantum stochastic process by introducing an effective Hamiltonian. From both approaches one can obtain a straightforward limit of a quantum system whose characteristic frequency is modulated stochastically. For the quantum system–bath case a nonlinear drift motion appears in the system along with the usual decoherence part which appears in both cases. When the system is a harmonic oscillator this nonlinear drift term is a Kerr nonlinear term. For the quantum bath case this nonlinearity will induce an extra diffusion mechanism along with classical phase diffusion.

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